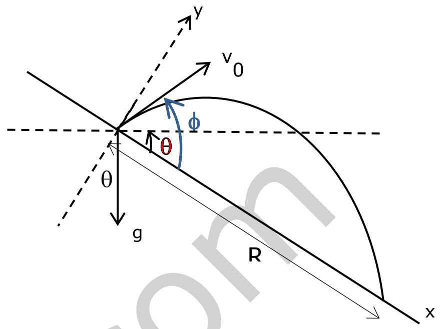
Q- Find the angle of projection φ with the incline to maximize the range of a projectile down the incline making an angle θ bellow horizontal and calculate this maximum range if the speed of projection is V_0.

Solution:



The incline makes angle – θ with the horizontal means it is bellow the horizontal (for down the incline) and the magnitude is θ.

Considering x axis down the incline and y axis normal to incline as in figure and resolving the velocity and acceleration of the projectile along the incline and normal to the incline

Velocity along the incline $v_x = v_0 \cos\phi$

Velocity normal to the incline $v_y = v_0 \sin\phi$

Acceleration along the incline $a_x = g \sin\theta$

Acceleration normal to the incline $a_y = - g \cos\theta$

Now if T is the time of flight before the projectile strikes the incline, than the displacement of the projectile normal to the incline will be zero and hence using second equation of motion we get

$[s = ut + \frac{1}{2} at^2]$

$0 = v_0 \sin\phi * T + \frac{1}{2}(- g \cos\theta) * T^2$

Gives $T = \frac{2v_0 sin\varphi}{g cos\theta}$ (As T cannot be zero) ------------- (1)

Now the range R on the incline is the displacement down the incline in this time T and hence given by

$[s = ut + \frac{1}{2} at^2]$

$$R = v_0 \cos\varphi\, T + \frac{1}{2} g \sin\theta\, T^2$$

Substituting value of T from equation (1) we get

$$R = v_0 \cos\varphi \left(\frac{2v_0 sin\varphi}{g cos\theta}\right) + \frac{1}{2} g \sin\theta \left(\frac{2v_0 sin\varphi}{g cos\theta}\right)^2$$

Or $$R = \frac{2{v_0}^2 sin\varphi}{g\, cos^2\theta}(cos\varphi\, cos\theta + sin\varphi\, sin\theta)$$

Or $$R = \frac{2{v_0}^2 sin\varphi \cos(\varphi - \theta)}{g\, cos^2\theta}$$ ------------- (2)

This range is a function of the angle ϕ and hence for it to be maximum or minimum dR/dϕ must be equal to zero

Thus, for R to be maximum

$$\frac{dR}{d\varphi} = \frac{d}{d\varphi}\left(\frac{2{v_0}^2 sin\varphi \cos(\varphi - \theta)}{g\, cos^2\theta}\right) = 0$$

Or $$\frac{d}{d\varphi}(sin\varphi \cos(\varphi - \theta)) = 0$$

Or $$- sin\varphi\, sin(\varphi - \theta) + \cos(\varphi - \theta)\, cos\varphi = 0$$

Or $$\cos(2\varphi - \theta) = \cos\frac{\pi}{2}$$

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Or $\cos(2\varphi - \theta) = 2n\pi \pm \left(\frac{\pi}{2}\right)$ (using general solution of cos ϕ = cos α)

Or $2\varphi - \theta = \left(\frac{\pi}{2}\right)$ (n=0 and for angel less then π)

Gives $\varphi = \left(\frac{\pi}{4} + \frac{\theta}{2}\right)$ ------------- (3)

This is the angle of projection with incline for maximum range of the projectile down the incline.

The range for any angle of projection ϕ is given in equation (2) and the condition for the range to be maximum i.e. the angle of projection ϕ for the maximum range down the incline is given by equation (3) hence for maximum range down the incline substituting ϕ in equation (2) from equation (3) we get

$$R_{max} = \frac{2{v_0}^2 sin\left(\frac{\pi}{4}+\frac{\theta}{2}\right)\cos\left(\frac{\pi}{4}+\frac{\theta}{2}-\theta\right)}{g\ cos^2\theta}$$

Or $$R_{max} = \frac{2{v_0}^2 sin\left(\frac{\pi}{4}+\frac{\theta}{2}\right)\cos\left(\frac{\pi}{4}-\frac{\theta}{2}\right)}{g\ cos^2\theta}$$

Using the trigonometric formula **2sinA cosB = sin (A+B) + sin (A - B)** we get

$$R_{max} = \frac{{v_0}^2\left[sin\left(\frac{\pi}{4}+\frac{\theta}{2}+\frac{\pi}{4}-\frac{\theta}{2}\right)+sin\left(\frac{\pi}{4}+\frac{\theta}{2}-\frac{\pi}{4}+\frac{\theta}{2}\right)\right]}{g\ cos^2\theta}$$

Or $$R_{max} = \frac{{v_0}^2\left[sin\left(\frac{\pi}{2}\right)+sin\ (\theta)\right]}{g\ cos^2\theta}$$

Or $$R_{max} = \frac{{v_0}^2(1+sin\theta)}{g\ cos^2\theta} = \frac{{v_0}^2(1+sin\theta)}{g\ (1-sin^2\theta)}$$

Or $$R_{max} = \frac{{v_0}^2(1+sin\theta)}{g\ (1-sin\ \theta)(1+sin\theta)}$$

Or $$R_{max} = \frac{{v_0}^2}{g\ (1-sin\theta)}$$

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