Q- A satellite of the Earth is moving in a circular orbit of radius R. By what fraction its velocity v be increased for the satellite to be in an elliptical orbit with minimum distance from the Earth equal to R and maximum distance from the Earth equal to 2R?

A satellite moves in circular orbit with constant speed only when the gravitational force on it due to earth is just sufficient to provide necessary centripetal force. Hence if the mass of the satellite be m and that of the earth M then equating magnitudes of the two forces in first situation we get

$$\frac{mv^2}{R} = \frac{GMm}{R^2}$$

Or $$\frac{GMm}{R} = mv^2 \qquad \text{---------------------- (1)}$$

When the satellite is in circular orbit of radius R its total energy is the sum of its gravitational potential energy and the kinetic energy thus gives the energy of the satellite in first situation.

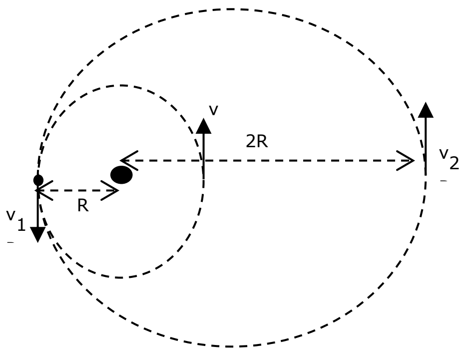
$$U_{initial} = PE + KE = -\frac{GMm}{R} + \frac{1}{2}mv^2$$

Substituting value from equation (1) we get

$$U_{initial} = -mv^2 + \frac{1}{2}mv^2 = -\frac{1}{2}mv^2 \qquad \text{---------------------- (2)}$$

Now let the rocket increase the velocity of the satellite suddenly from v to v_1. Then the total energy of the satellite just after the rocket is fired will become

$$U_{final} = -\frac{GMm}{R} + \frac{1}{2}m{v_1}^2 \qquad \text{---------------------- (3)}$$



With increase in the velocity of the satellite the centripetal force required to move it on the circular orbit (mv^2/R) will be more and the gravitational force of the earth is now not sufficient to provide this force and hence the orbit changes from circular to the elliptical orbit.

As in elliptic orbit the gravitational force is always along the center of earth, net torque on the satellite due to earth's gravitational force is always zero and hence its angular momentum about the center of earth will remain conserved.

Let the velocity of the satellite at the other extreme (at distance r_{max} = 2R) is v_2. As the angular momentum is the moment of momentum we get from law of conservation of angular momentum

$$mv_1 * R = mv_2 * 2R$$

And hence the speed of the satellite at farthest point will be

$$v_2 = \frac{1}{2}v_1 \qquad \text{------------------------- (4)}$$

Now neglecting resistance if any the total energy of the satellite will be conserved and hence considering energy at nearest and farthest point, we get

$$U_{final} = PE + KE = -\frac{GMm}{R} + \frac{1}{2}m{v_1}^2 = -\frac{GMm}{2R} + \frac{1}{2}m{v_2}^2$$

Or $-\frac{GMm}{R} + \frac{1}{2} m{v_1}^2 = -\frac{GMm}{2R} + \frac{1}{2} m{v_2}^2$

Substituting value of v_2 from equation (4) we get

$$-\frac{GMm}{R} + \frac{1}{2} m{v_1}^2 = -\frac{GMm}{2R} + \frac{1}{2} m\left(\frac{v_1}{2}\right)^2$$

Or $\frac{1}{2} m{v_1}^2 - \frac{1}{2} m\left(\frac{v_1}{2}\right)^2 = -\frac{GMm}{2R} + \frac{GMm}{R}$

Or $\frac{3}{4} m{v_1}^2 = \frac{GMm}{R}$

And again, from equation (1)

$$\frac{3}{4} m{v_1}^2 = mv^2$$

Or ${v_1}^2 = \frac{4}{3} v^2$

Gives $v_1 = \frac{2v}{\sqrt{3}}$

Thus, the fraction by which the velocity should be increased be

$$f = \frac{v_1 - v}{v} = \frac{2}{\sqrt{3}} - 1 = \frac{2-\sqrt{3}}{\sqrt{3}} = 0.155$$