

Q- A spherical capacitor has two different layers of dielectrics between its spheres. Their permittivity is ϵ_1 for $a < r < r_0$ and ϵ_2 for $r_0 < r < b$. Find the capacitance of this system by finding the total energy of the fields between the spheres.

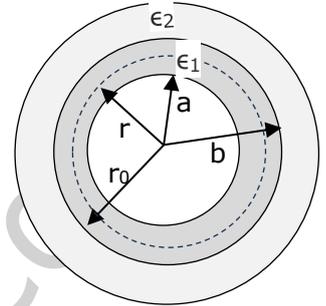
Solution:

The energy of a capacitor is due to its field energy. Wherever there is an electric field an energy is associated with it. The energy associated with an electric field E is given by

$$U = \frac{1}{2} \epsilon E^2 V$$

Here is the permittivity and V is the volume.

As the field here is varying with the radius, we have to calculate the field using integral calculus.



Consider a very thin spherical shell of radius r and thickness dr . Volume of this shell will be $4\pi r^2 dr$. If Q is the charge on the inner surface of the capacitor field at distance r will be

$$E = \frac{Q}{4\pi\epsilon r^2}$$

Now if r is between a and r_0 the energy associated with the shell will be

$$dU = \frac{1}{2} \epsilon_1 \left(\frac{Q}{4\pi\epsilon_1 r^2} \right)^2 4\pi r^2 dr$$

And hence the energy stored in the inner layer of dielectric will be

$$U_1 = \int dU = \frac{1}{2} \epsilon_1 \left(\frac{Q}{4\pi\epsilon_1} \right)^2 4\pi \int_a^{r_0} \frac{dr}{r^2} = \frac{Q^2}{8\pi\epsilon_1} \left(\frac{1}{a} - \frac{1}{r_0} \right)$$

Similarly, the energy stored in the second layer is given by

$$U_2 = \int dU = \frac{1}{2} \epsilon_2 \left(\frac{Q}{4\pi\epsilon_2} \right)^2 4\pi \int_{r_0}^b \frac{dr}{r^2} = \frac{Q^2}{8\pi\epsilon_2} \left(\frac{1}{r_0} - \frac{1}{b} \right)$$

And thus, total energy stored in the capacitor will be

$$U = U_1 + U_2 = \frac{Q^2}{8\pi\epsilon_1} \left(\frac{1}{a} - \frac{1}{r_0} \right) + \frac{Q^2}{8\pi\epsilon_2} \left(\frac{1}{r_0} - \frac{1}{b} \right)$$

Now as the energy stored in a capacitor is given by $U = Q^2/2C$, we get

$$C = Q^2/2U$$

$$\text{Or } C = \frac{Q^2}{2 \left[\frac{Q^2}{8\pi\epsilon_1} \left(\frac{1}{a} - \frac{1}{r_0} \right) + \frac{Q^2}{8\pi\epsilon_2} \left(\frac{1}{r_0} - \frac{1}{b} \right) \right]} = \frac{4\pi}{\frac{1}{\epsilon_1} \left(\frac{1}{a} - \frac{1}{r_0} \right) + \frac{1}{\epsilon_2} \left(\frac{1}{r_0} - \frac{1}{b} \right)}$$

Alternative Method:

The potential difference between the two spheres is given by

$$V = \int E \cdot dr = \frac{Q}{4\pi} \left[\frac{1}{\epsilon_1} \int_a^{r_0} \frac{dr}{r^2} + \frac{1}{\epsilon_2} \int_{r_0}^b \frac{dr}{r^2} \right] = \frac{Q}{4\pi} \left[\frac{1}{\epsilon_1} \left(\frac{1}{a} - \frac{1}{r_0} \right) + \frac{1}{\epsilon_2} \left(\frac{1}{r_0} - \frac{1}{b} \right) \right]$$

Using $C = Q/V$ will give the same result.

(Here ϵ is permittivity not relative permittivity and neglecting charge on the outermost surface)